

A STRONG CENTRAL LIMIT THEOREM FOR A CLASS OF RANDOM SURFACES

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ABSTRACT. This paper is concerned with $d = 2$ dimensional lattice field models with action $V(\nabla\phi(\cdot))$, where $V : \mathbf{R}^d \rightarrow \mathbf{R}$ is a uniformly convex function. The fluctuations of the variable $\phi(0) - \phi(x)$ are studied for large $|x|$ via the generating function given by $g(x, \mu) = \ln \langle e^{\mu(\phi(0) - \phi(x))} \rangle_A$. In two dimensions $g''(x, \mu) = \partial^2 g(x, \mu) / \partial \mu^2$ is proportional to $\ln |x|$. The main result of this paper is a bound on $g'''(x, \mu) = \partial^3 g(x, \mu) / \partial \mu^3$ which is uniform in $|x|$ for a class of convex V . The proof uses integration by parts following Helffer-Sjöstrand and Witten, and relies on estimates of singular integral operators on weighted Hilbert spaces.

1. INTRODUCTION.

We shall be interested in probability spaces (Ω, \mathcal{F}, P) associated with certain Euclidean lattice field theories. These Euclidean field theories are determined by a potential $V : \mathbf{R}^d \rightarrow \mathbf{R}$ which is a C^2 uniformly convex function. Thus the second derivative $V''(\cdot)$ of $V(\cdot)$ is assumed to satisfy the quadratic form inequality

$$(1.1) \quad \lambda I_d \leq V''(z) \leq \Lambda I_d, \quad z = (z_1, \dots, z_d) \in \mathbf{R}^d,$$

where I_d is the identity matrix in d dimensions and λ, Λ are positive constants. The measure P is formally given as

$$(1.2) \quad P = \exp \left[- \sum_{x \in \mathbf{Z}^d} V(\nabla\phi(x)) \right] \prod_{x \in \mathbf{Z}^d} d\phi(x) / \text{normalization},$$

where ∇ is the discrete gradient operator acting on fields $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$. In the case when $V(z) = |z|^2/2 + a \sum_{j=1}^d \cos z_j$, $z \in \mathbf{R}^d$, the probability measure (1.2) describes the dual representation of a gas of lattice dipoles with activity a (see [4]).

We denote the adjoint of ∇ by ∇^* . Thus ∇ is a d dimensional *column* operator and ∇^* a d dimensional *row* operator, which act on functions $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$ by

$$(1.3) \quad \begin{aligned} \nabla\phi(x) &= (\nabla_1\phi(x), \dots, \nabla_d\phi(x)), & \nabla_i\phi(x) &= \phi(x + \mathbf{e}_i) - \phi(x), \\ \nabla^*\phi(x) &= (\nabla_1^*\phi(x), \dots, \nabla_d^*\phi(x)), & \nabla_i^*\phi(x) &= \phi(x - \mathbf{e}_i) - \phi(x). \end{aligned}$$

In (1.3) the vector $\mathbf{e}_i \in \mathbf{Z}^d$ has 1 as the i th coordinate and 0 for the other coordinates, $1 \leq i \leq d$. Note that the Hessian of our action in (1.2) is a uniformly elliptic finite difference operator acting on $\ell^2(\mathbf{Z}^d)$ given by

$$\nabla^* V''(\nabla\phi(\cdot)) \nabla.$$

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Let Ω be the space of all functions $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$ and \mathcal{F} be the Borel algebra generated by finite dimensional rectangles $\{\phi \in \Omega : |\phi(x_i) - a_i| < r_i, i = 1, \dots, N\}$, $x_i \in \mathbf{Z}^d$, $a_i \in \mathbf{R}$, $r_i > 0$, $i = 1, \dots, N$, $N \geq 1$. The d dimensional integer lattice \mathbf{Z}^d acts on Ω by translation operators $\tau_x : \Omega \rightarrow \Omega$, $x \in \mathbf{Z}^d$, where $\tau_x \phi(z) = \phi(x + z)$, $z \in \mathbf{Z}^d$. Translation operators are measurable and satisfy the properties $\tau_x \tau_y = \tau_{x+y}$, $\tau_0 = \text{identity}$, $x, y \in \mathbf{Z}^d$. It was first shown by Funaki and Spohn [6] that one can define a unique ergodic translation invariant probability measure P on (Ω, \mathcal{F}) corresponding to (1.2). If $d \geq 3$ this is a measure on fields $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$, but for $d = 1, 2$, one needs to regard (1.2) as a measure on gradient fields $\omega = \nabla \phi$. In that case the Borel algebra \mathcal{F} is generated by finite dimensional rectangles for $\omega(\cdot)$ with the usual gradient constraint that the sum of $\omega(\cdot)$ over plaquettes is zero.

Estimates on expectation values $\langle \cdot \rangle_\Omega$ for (Ω, \mathcal{F}, P) can be obtained from the Brascamp-Lieb inequality [2]. Since by (1.1) we have a uniform lower bound on the Hessian, this inequality implies that for $f : \mathbf{Z}^d \rightarrow \mathbf{R}$, with $\sum_{y \in \mathbf{Z}^d} f(y) = 0$

$$(1.4) \quad \langle \exp[(f, \phi)] \rangle_\Omega \leq \exp \left[\frac{1}{2} (f, (-\lambda \Delta)^{-1} f) \right],$$

where (\cdot, \cdot) denotes the standard inner product for functions on \mathbf{Z}^d and Δ is the discrete Laplacian on \mathbf{Z}^d . Note that we have used translation invariance so that $\langle (\phi, f) \rangle = 0$. The inequality (1.4) can be written in gradient form: for $h : \mathbf{Z}^d \rightarrow \mathbf{R}^d$ one has

$$(1.5) \quad \langle \exp[(h, \omega)] \rangle_\Omega \leq \exp [\|h\|^2 / 2\lambda],$$

where $\omega(\cdot) = \nabla \phi(\cdot)$ and $\|\cdot\|$ denotes the L^2 norm. It follows from (1.4) that the function $g(\cdot, \cdot)$ defined by

$$(1.6) \quad g(x, \mu) = \log \langle e^{\mu(\phi(0) - \phi(x))} \rangle_\Omega, \quad \mu \in \mathbf{R}$$

satisfies the inequality $g(x, \mu) \leq C_d \mu^2$ for some constant C_d provided $d \geq 3$. If $d = 1, 2$ then (1.5) implies that $g(x, \mu) \leq C_d(x) \mu^2$ where $C_2(x) \sim \log |x|$ and $C_1(x) \sim |x|$ for large $|x|$. Since in dimension $d = 1$ the random variables $\nabla \phi(x)$, $x \in \mathbf{Z}$, are i.i.d., it is easy to see that in this case $g(x, \mu) = C(\mu)|x|$ for a positive constant $C(\mu)$ depending only on μ . In this paper we shall show that the x dependence of $C_2(x)$ for large $|x|$ is entirely due to the second moment of $\phi(x) - \phi(0)$.

Theorem 1.1. *Suppose $d = 2$ and $V : \mathbf{R}^d \rightarrow \mathbf{R}$ is C^3 , satisfies the inequality (1.1) and $\|V'''(\cdot)\|_\infty = M < \infty$. If in addition $\lambda/\Lambda > 1/2$, then there is a positive constant C depending only on λ, Λ , such that*

$$(1.7) \quad |g'''(x, \mu) = \frac{\partial^3 g(x, \mu)}{\partial \mu^3}| \leq CM \quad x \in \mathbf{Z}^d, \mu \in \mathbf{R}.$$

Hence we have $|g(x, \mu) - \frac{\mu^2}{2} \langle (\phi(0) - \phi(x))^2 \rangle_\Omega| \leq C\mu^3 M/6$, $x \in \mathbf{Z}^d$.

Remark: If $(\phi(0) - \phi(x))$ is Gaussian then $g'''(x, \mu) = 0$. Note that in one dimension, $g'''(x, \mu) \propto |x|$ unless our distribution is Gaussian. Thus the analog of our theorem is *not* valid in one dimension. In this sense, the long range correlation of the gradient fields in 2D give a stronger CLT.

The proof of Theorem 1.1 follows from an inequality for the third moment of $\phi(0) - \phi(x)$,

$$(1.8) \quad \frac{\partial^3 g(x, \mu)}{\partial \mu^3} = \langle [X - \langle X \rangle_{\Omega, x, \mu}]^3 \rangle_{\Omega, x, \mu}, \quad \text{where } X = \phi(0) - \phi(x),$$

and $\langle \cdot \rangle_{\Omega, x, \mu}$ denotes expectation with respect to the probability measure proportional to

$$(1.9) \quad e^{\mu(\phi(0) - \phi(x))} dP(\phi(\cdot)) ,$$

with P the translation invariant measure (1.2). If $\mu = 0$ and the function $V(\cdot)$ of (1.2) is symmetric i.e. $V(z) = V(-z)$, $z \in \mathbf{R}^d$, then it is easy to see that the third moment of $\phi(0) - \phi(x)$ is 0. More generally we have the following decay estimate:

Theorem 1.2. *Under the assumptions of Theorem 1.1 we have:*

$$(1.10) \quad | \langle (\phi(0) - \phi(x))^3 \rangle_{\Omega} | \leq CM/[1 + |x|^{\alpha}], \quad x \in \mathbf{Z}^d ,$$

for some positive α .

Relation to dimers: In two dimensions one can think of $\phi(x)$, $x \in \mathbf{Z}^2$, as being the height of a random surface over \mathbf{Z}^2 which fluctuates logarithmically. Theorem 1.1 was motivated by related results for dimer models. The uniform measure on dimer covers of the square lattice has an associated height function $\phi(\cdot)$ which takes *integer* values. Denoting by $\langle \cdot \rangle_D$ the expectation on heights induced by the uniform measure on dimers, the height fluctuations $\langle (\phi(0) - \phi(x))^2 \rangle_D$ grow logarithmically with $|x|$ (see [9, 10] for an introduction to dimers and heights). As in Euclidean field theory with measure (1.2), one can consider the function $g(x, \mu)$ defined by

$$(1.11) \quad g(x, \mu) = \log \langle e^{\mu(\phi(0) - \phi(x))} \rangle_D , \quad x \in \mathbf{Z}^2 ,$$

but in this case it is interesting to let μ be *pure imaginary*, whereas in Theorem 1.1 μ is *real*. In [13] it is shown that there exists $\delta > 0$ such that

$$(1.12) \quad | g(x, \mu) - \frac{\mu^2}{2} \langle (\phi(0) - \phi(x))^2 \rangle_D | \leq C, \quad x \in \mathbf{Z}^2, \mu \in i\mathbf{R}, |\mu| < \delta,$$

for some constant C . This implies that $\langle e^{\mu(\phi(0) - \phi(x))} \rangle_D$ has a power law decay which is determined only by the variance. Since one also has [9, 10] that

$$(1.13) \quad \langle (\phi(0) - \phi(x))^2 \rangle_D = \frac{16}{\pi^2} \log |x| + O(1) \quad \text{as } |x| \rightarrow \infty,$$

the inequality (1.12) gives rather precise information on the behavior of $\langle e^{\mu(\phi(0) - \phi(x))} \rangle_D$ for large $|x|$ and small μ . The inequality (1.12) for x lying along lattice lines follows from earlier work [1] on Toeplitz determinants for piecewise smooth symbols. These results allow for a larger range of δ in (1.12) than [13] does. In special cases this power law decay is related to the spin-spin correlation of an Ising antiferromagnet on a triangular lattice at 0 temperature. Note that since the heights are integer valued, when $\mu = 2\pi i$, $g(x, \mu) \equiv 0$ so that (1.12) cannot hold for all μ .

A closely related central limit theorem arises in fluctuations of the number of eigenvalues of a $U(N)$ matrix belonging to an arc on the circle. The variance of this number grows logarithmically in N . If we call the corresponding generating function $g(N, \mu)$, then for a suitable range of μ we have $|g'''(N, \mu)| \leq \text{Constant}$. If the indicator function of the arc is smoothed out, the logarithmic growth in N disappears. Note that the methods of this paper do not apply to dimer heights or to $U(N)$ because the integer constraints make the associated action non-convex.

Idea of Proof: The reason that a stronger form of CLT holds in dimension 2 may be understood as follows: We can express

$$(1.14) \quad \phi(0) - \phi(x) = \sum_{y \in \mathbf{Z}^d} \nabla \phi(y) \cdot [\nabla G_0(y) - \nabla G_0(y - x)] ,$$

where G_0 is the Green's function of the discrete Laplacian. In 2D the sum of the gradients is spread out since $|\nabla G_0(y)| \sim (|y| + 1)^{-1}$. Although this function is not square summable, it lies in the weighted ℓ^2 space $\ell_w^2(\mathbf{Z}^2, \mathbf{R}^2)$ with weight $w(y) = [1 + |y|]^\alpha$ for any $\alpha < 0$. Hence from the theory of singular integral operators [15] the convolution of $\nabla \nabla^* G_0(\cdot)$ with $\nabla G_0(\cdot)$ is also in the space $\ell_w^2(\mathbf{Z}^2, \mathbf{R}^2)$. This situation should be contrasted with the case of one dimension where the gradient has no decay.

In order to implement our argument, which is based on the intuition gained from (1.14) and the decay of the 2D Green's function, we use an integration by parts formula due to Helffer-Sjöstrand and Witten [7, 8], and some results on singular integral operators on weighted spaces. The integration by parts formula can be stated formally as

$$(1.15) \quad \langle (F_1(\cdot) - \langle F_1 \rangle) F_2(\cdot) \rangle_{\Omega, x, \mu} = \langle dF_1(\cdot) \cdot [d^* d + \nabla^* V''(\nabla \phi(\cdot)) \nabla]^{-1} dF_2(\cdot) \rangle_{\Omega, x, \mu}.$$

In (1.15) expectation is with respect to the measure (1.9), and the functions $F_i(\phi(\cdot))$ are differentiable functions of the field $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$. The operator d is the gradient operator acting on functions of $\phi(\cdot)$, and $d^* = -d + \nabla V + \mu \nabla X$ is the corresponding divergence operator with respect to the measure (1.9). Hence $d^* d$ is the elliptic self-adjoint operator acting on functions of $\phi(\cdot)$, which corresponds to the Dirichlet form for (1.9). The identity (1.15) is explained in more detail in the following section, and since $d^* d$ is nonnegative it implies (1.4). The operator $d^* d + \nabla^* V''(\nabla \phi(\cdot)) \nabla$ formally acts on functions $F(y, \phi(\cdot))$. The first term acts as a differential operator in the field variable $\phi(\cdot)$, and the second term acts as a finite difference elliptic operator in the lattice index y .

We first prove an L^2 version of Theorem 1.1 using the integration by parts formula (1.15). This result unfortunately requires the seemingly artificial restriction $\lambda/\Lambda > 1/2$ on the bounds (1.1). The reason for the restriction on λ/Λ is that we need to express our Green's function so that second order finite difference derivatives ∇_x appear in a symmetric way. This problem arises due to the presence of the operator $d^* d$, and therefore does not occur in the classical case where we set $d^* d \equiv 0$. See Lemma 2.2 and the resolvent expansion for (2.33).

Theorems 1.1 and 1.2 follow from an extension of the L^2 theorem to the corresponding theorem for weighted L^2 spaces, with weights which are in the Muckenhoupt A_2 class [15]. The weights can be chosen arbitrarily close to the constant function in the A_2 norm, and so Theorem 1.1 also holds with the restriction $\lambda/\Lambda > 1/2$. The reason for this is that the norm of a Calderon-Zygmund operator on an A_p weighted space is a continuous function of the A_p norm at the constant. This continuity result does not follow from the standard proofs [15] of the boundedness of Calderon-Zygmund operators on weighted spaces, and was proven quite recently [12]. If one however restricts to weights which are dilation and rotation invariant, continuity follows from the argument in a classical paper on the subject [14]. The weights considered in this paper are approximately rotation and dilation invariant.

In §2 we first state and prove (1.15) and then obtain an estimate on the third moment of $\sum_x h(x) \nabla \phi(x)$ with h in $\ell_2(\mathbf{Z}^2)$. There are two proofs for the third moment. The first proof uses quadratic form inequalities, and the second uses a convergent perturbation expansion. In §4 it is shown that the perturbation expansion also converges for functions in weighted spaces with weight close to 1. As explained above this is needed to prove Theorem 1.1 since $(\phi(0) - \phi(x)) = \sum_x h(x) \nabla \phi(x)$ with h

in a weighted ℓ_2 space. See (1.14). The required weighted norm inequalities for functions on \mathbf{Z}^d are proved in §3. These inequalities are applied to the field theory setting in §4 by using the spectral decomposition of the self-adjoint operator d^*d . Because we need to make use of the spectral decomposition theorem, we cannot replace the weighted norm inequalities in our argument by L^q inequalities with q close to 2.

2. THE L^2 THEORY

Our main goal in this section will be to establish an L^2 version of Theorem 1.1. First we shall state and prove a finite dimensional version of the Helffer-Sjöstrand formula (1.15) which we shall use in the proof.

Let L be a positive even integer and $Q = Q_L \subset \mathbf{Z}^d$ be the integer lattice points in the cube centered at the origin with side of length L . By a periodic function $\phi : Q \rightarrow \mathbf{R}$ we mean a function ϕ on Q with the property that $\phi(x) = \phi(y)$ for all $x, y \in Q$ such that $x - y = L\mathbf{e}_k$ for some k , $1 \leq k \leq d$. Let Ω_Q be the space of all periodic functions $\phi : Q \rightarrow \mathbf{R}$, whence Ω_Q with $Q = Q_L$ can be identified with \mathbf{R}^N where $N = L^d$. Let \mathcal{F}_Q be the Borel algebra for Ω_Q which is generated by the open sets of \mathbf{R}^N . For $m > 0$, we define a probability measure $P_{Q,m}$ on $(\Omega_Q, \mathcal{F}_Q)$ as follows:

$$(2.1) \quad \langle F(\cdot) \rangle_{\Omega_Q, m} = \int_{\mathbf{R}^N} F(\phi(\cdot)) \exp \left[- \sum_{x \in Q} \left\{ V(\nabla \phi(x)) + \frac{1}{2} m^2 \phi(x)^2 \right\} \right] \prod_{x \in Q} d\phi(x) / \text{normalization} ,$$

where $F : \mathbf{R}^N \rightarrow \mathbf{R}$ is a continuous function such that $|F(z)| \leq C \exp[A|z|]$, $z \in \mathbf{R}^N$, for some constants C, A . Note that $\langle \phi(x) \rangle_{\Omega_Q, m} = 0$ for all $x \in Q$. This follows from the translation invariance of the measure (2.1), upon making the change of variable $\phi(\cdot) \rightarrow \phi(\cdot) + \varepsilon$, differentiating with respect to ε and setting $\varepsilon = 0$. We consider now for $\mu \in \mathbf{R}$ and $x \in Q$ the probability measure proportional to the measure

$$(2.2) \quad e^{\mu(\phi(0) - \phi(x))} dP_{Q,m}(\phi(\cdot))$$

on $(\Omega_Q, \mathcal{F}_Q, P_{Q,m})$, which is analogous to (1.9), and denote expectation with respect to this measure by $\langle \cdot \rangle_{\Omega_Q, m, x, \mu}$. Let $F : \mathbf{R}^N \rightarrow \mathbf{R}$ be a C^1 function and $dF : \mathbf{R}^N \rightarrow \mathbf{R}^N$ be its gradient. For a C^1 function $G : \mathbf{R}^N \rightarrow \mathbf{R}^N$ the divergence d^*G of G with respect to the measure (2.2) is formally defined from the integration by parts formula

$$(2.3) \quad \langle (G, dF) \rangle_{\Omega_Q, m, x, \mu} = \langle (d^*G, F) \rangle_{\Omega_Q, m, x, \mu} .$$

Lemma 2.1 (Helffer-Sjöstrand). *Let F_1, F_2 be two C^1 functions on \mathbf{R}^N such that for $j = 1, 2$, the inequality $|F_j(z)| + |DF_j(z)| \leq C \exp[A|z|]$, $z \in \mathbf{R}^N$, holds for some constants C, A . If $\langle F_2(\cdot) \rangle_{\Omega_Q, m, x, \mu} = 0$ then there is the identity*

$$(2.4) \quad \langle F_1(\cdot) F_2(\cdot) \rangle_{\Omega_Q, m, x, \mu} = \langle dF_1(\cdot) [d^*d + \nabla^* V''(\nabla \phi(\cdot)) \nabla + m^2]^{-1} dF_2(\cdot) \rangle_{\Omega_Q, m, x, \mu} .$$

Proof. We first assume that the Poincaré inequality

$$(2.5) \quad \langle F(\cdot)^2 \rangle \leq K \langle |dF(\cdot)|^2 \rangle ,$$

holds for the linear space \mathcal{E} of C^1 functions $F : \mathbf{R}^N \rightarrow \mathbf{R}$ satisfying $|F(z)| + |DF(z)| \leq C \exp[A|z|]$, $z \in \mathbf{R}^N$, for some constants C, A and $\langle F(\cdot) \rangle = 0$. Next we define the Hilbert space \mathcal{H} as the closure of \mathcal{E} under the norm $\|\cdot\|$ defined by $\|F\|^2 = \langle |F(\cdot)|^2 \rangle$. The Hilbert space \mathcal{H}_{grad} is defined as the closure of the linear space $\{G(\cdot) = dF(\cdot) : F(\cdot) \in \mathcal{E}\}$ under the norm $\|G(\cdot)\|^2 = \langle |G(\cdot)|^2 \rangle$.

Since $F_2(\cdot) \in \mathcal{H}$, it follows from (2.5) that there exists a solution $F_3(\cdot) \in \mathcal{H}$ with $dF_3(\cdot) \in \mathcal{H}_{grad}$ to the equation

$$(2.6) \quad d^*dF_3(\cdot) = F_2(\cdot), \quad \text{implies } \langle F_1(\cdot)F_2(\cdot) \rangle = \langle dF_1(\cdot)dF_3(\cdot) \rangle.$$

If we assume that $F_2(\cdot)$ is a $C^{1+\alpha}$ function for any $\alpha > 0$ then elliptic regularity theory implies that $F_3(\cdot)$ is C^3 . The identity (2.4) follows then from (2.6) by observing that

$$(2.7) \quad dF_2(\cdot) = (dd^*)dF_3(\cdot) = [d^*d + \nabla^*V''(\nabla\phi(\cdot))\nabla + m^2]dF_3(\cdot).$$

Note that above we have used the fact that the commutator $[d^*, d]$ is the Hessian.

Since $C^{1+\alpha}$ functions $F(\cdot) \in \mathcal{E}$ are dense in \mathcal{H} and their gradients $dF(\cdot)$ are dense in \mathcal{H}_{grad} , the result follows provided (2.5) holds.

In order to remove the assumption (2.5) we note that (2.4) implies that (2.5) holds with $K = 1/m^2$. It is clear then that one can prove the Poincaré inequality (2.5) by arguing as in the previous paragraph but replacing $F_3(\cdot)$ by $F_{3,\varepsilon}(\cdot)$ where $[d^*d + \varepsilon]F_{3,\varepsilon}(\cdot) = F_2(\cdot)$ and letting $\varepsilon \rightarrow 0+$. \square

Remark: For periodic $f : Q \rightarrow \mathbf{R}$ let $g_{Q,m}(x, \mu, f(\cdot))$ be defined by

$$(2.8) \quad g_{Q,m}(x, \mu, f(\cdot)) = \log\{\langle \exp[\mu(f, \phi)] \rangle_{\Omega_{Q,m}}\}.$$

Then since $g_{Q,m}(x, 0, f(\cdot)) = \partial g_{Q,m}(x, 0, f(\cdot))/\partial \mu = 0$, and $\partial^2 g_{Q,m}(x, \mu, f(\cdot))/\partial \mu^2$ has the form (2.4) with $F_i(\phi(\cdot)) = (f, \phi)$, one obtains the Brascamp-Lieb inequality

$$(2.9) \quad \langle \exp[(f, \phi)] \rangle_{\Omega_{Q,m}} \leq \exp \left[\frac{1}{2} (f, \{-\lambda\Delta + m^2\}^{-1} f) \right],$$

which is a finite dimensional version of the inequality (1.4). Evidently the function $g(x, \mu)$ of (1.6) corresponds to $f(\cdot) = \delta_0(\cdot) - \delta_x(\cdot)$ in (2.8).

The measure (1.2) can be constructed [5, 6] as a limit of measures (2.1) by letting first $Q \rightarrow \mathbf{Z}^d$ and then $m \rightarrow 0$. The probability space $(\Omega, \mathcal{F}, P_m)$ on fields $\phi : \mathbf{Z}^d \rightarrow \mathbf{R}$ is obtained as the limit of the spaces $(\Omega_Q, \mathcal{F}_Q, P_{Q,m})$ as $|Q| \rightarrow \infty$. In particular one has from Lemma 2.2 of [5] the following result:

Proposition 2.1. *Assume $m > 0$ and let $F : \mathbf{R}^k \rightarrow \mathbf{R}$ be a C^1 function which satisfies the inequality*

$$(2.10) \quad |DF(z)| \leq A \exp[B|z|], \quad z \in \mathbf{R}^k,$$

for some constants A, B . Then for any $x_1, \dots, x_k \in \mathbf{Z}^d$, the limit

$$(2.11) \quad \lim_{|Q| \rightarrow \infty} \langle F(\phi(x_1), \phi(x_2), \dots, \phi(x_k)) \rangle_{\Omega_{Q,m}} = \langle F(\phi(x_1), \phi(x_2), \dots, \phi(x_k)) \rangle_{\Omega,m}$$

exists and is finite.

Proposition 2.1 enables us to define via the Helly-Bray theorem [3] the probability measure P_m on Ω by setting expectation values to be given by the limit (2.11). Evidently P_m is invariant under translations, and from the Brascamp-Lieb inequality we see that P_m is ergodic.

The probability space (Ω, \mathcal{F}, P) on gradient fields $\omega : \mathbf{Z}^d \rightarrow \mathbf{R}^d$ is obtained as the limit of the spaces $(\Omega, \mathcal{F}, P_m)$ as $m \rightarrow 0$. From the proof of Lemma 2.1 of [5] we have the following result:

Proposition 2.2. *Let $G : \mathbf{R}^{kd} \rightarrow \mathbf{R}$ be a C^1 function which satisfies the inequality*

$$(2.12) \quad |G(z)| \leq A \exp[B|z|], \quad z \in \mathbf{R}^{kd},$$

for some constants A, B . Then for any $x_1, \dots, x_k \in \mathbf{Z}^d$, the limit

$$(2.13) \quad \lim_{m \rightarrow 0} \langle G(\omega(x_1), \omega(x_2), \dots, \omega(x_k)) \rangle_{\Omega, m} = \langle G(\omega(x_1), \omega(x_2), \dots, \omega(x_k)) \rangle_{\Omega}$$

exists and is finite.

Just as with Proposition 2.1, we see that Proposition 2.2 enables us to define a measure corresponding to (1.2) which is translation invariant and ergodic. In the case $d = 1$ the variables $\omega(x)$, $x \in \mathbf{Z}$, are independent with density given by $\exp[-V(\omega(x)) - \rho\omega(x)]d\omega(x)/\text{normalization}$, where $\rho \in \mathbf{R}$ is the unique number such that the expectation of $\omega(x)$ is zero. For $d \geq 2$ the variables $\omega(x)$, $x \in \mathbf{Z}^d$, are correlated.

Theorem 2.1. *Suppose $d \geq 1$ and the constants in (1.1) satisfy $\lambda/\Lambda > 1/2$. Then there is a positive constant $C(\lambda, \Lambda)$ depending only on λ, Λ such that for any $h_1, h_2, h_3 \in \ell^2(\mathbf{Z}^d, \mathbf{R}^d)$ and $x \in \mathbf{Z}^d$, $\mu \in \mathbf{R}$,*

$$(2.14) \quad \left| \left\langle \prod_{j=1}^3 [h_j, \omega] - \langle (h_j, \omega) \rangle_{\Omega, x, \mu} \right\rangle_{\Omega, x, \mu} \right| \leq C(\lambda, \Lambda) \|h_1\| \|h_2\| \|h_3\| \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)|.$$

The proof of Theorem 2.1 depends on a representation for the third moment of (h, ω) , which we obtain by applying the Helffer-Sjöstrand formula of Lemma 2.1. We first obtain the representation for a periodic cube in \mathbf{Z}^d and then show using Proposition 2.1 and 2.2 that the representation continues to be valid as the cube increases to \mathbf{Z}^d .

Let $h_j : Q \rightarrow \mathbf{R}^d$, $j = 1, 2, 3$ be arbitrary periodic functions and define $G_j(\phi(\cdot))$ in terms of them by

$$(2.15) \quad G_j(\phi(\cdot)) = [(h_j(\cdot), \nabla \phi(\cdot)) - \langle (h_j(\cdot), \nabla \phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu}].$$

Applying (2.4) to the functions $F_1(\phi(\cdot)) = G_1(\phi(\cdot))G_2(\phi(\cdot))$ and $F_2(\phi(\cdot)) = G_2(\phi(\cdot))G_3(\phi(\cdot))$ yields the identity

$$(2.16) \quad \langle G_1(\phi(\cdot))G_2(\phi(\cdot))G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu} = \langle ([G_1(\phi(\cdot))\nabla^* h_2(\cdot) + G_2(\phi(\cdot))\nabla^* h_1(\cdot)], \Phi_3(\cdot, \phi(\cdot))) \rangle_{\Omega_Q, m, x, \mu},$$

where $\Phi_j(y, \phi(\cdot))$, $y \in Q$, $\phi(\cdot) \in \Omega_Q$, $j = 1, 2, 3$ is the solution to the equation

$$(2.17) \quad [d^*d + \nabla^* V''(\nabla \phi(y))\nabla + m^2] \Phi_j(y, \phi(\cdot)) = \nabla^* h_j(y), \quad y \in Q.$$

Since for each $y \in Q$ the expectation $\langle [G_1(\phi(\cdot))\nabla^* h_2(y) + G_2(\phi(\cdot))\nabla^* h_1(y)] \rangle_{\Omega_Q, m, x, \mu} = 0$, we can apply (2.4) again to the RHS of (2.16). Thus we obtain the identity

$$(2.18) \quad \langle G_1(\phi(\cdot))G_2(\phi(\cdot))G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu} = \sum_{y, z \in Q} \langle [\Phi_1(z, \phi(\cdot))\nabla^* h_2(y) + \Phi_2(z, \phi(\cdot))\nabla^* h_1(y)] d\Phi_3(y, z, \phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu},$$

where $d\Phi_j(y, z, \phi(\cdot))$, $z \in Q$, is the gradient of the function $\Phi_j(y, \phi(\cdot))$ which is the solution to (2.17). Since $\Phi_j(\cdot, \phi(\cdot))$ itself is the gradient of a function of $\phi(\cdot)$ it follows that $d\Phi_j(y, z, \phi(\cdot))$ is symmetric in (y, z) . By applying d to (2.17), it is easy to see that $d\Phi_j(y, z, \phi(\cdot))$ is the solution to the equation

$$(2.19) \quad \begin{aligned} & \sum_{y, z \in Q} f_1(y) f_2(z) [d^* d + \nabla_y^* V''(\nabla \phi(y)) \nabla_y + \nabla_z^* V''(\nabla \phi(z)) \nabla_z + 2m^2] d\Phi_j(y, z, \phi(\cdot)) \\ &= - \sum_{y, z \in Q} V'''(\nabla \phi(y)) [\nabla f_1(y), \nabla f_2(z), \nabla \Phi_j(y, \phi(\cdot))] \delta(y - z), \quad f_1, f_2 : Q \rightarrow \mathbf{R}, \end{aligned}$$

where $V'''(\xi)[\cdot, \cdot, \cdot]$ denotes the symmetric trilinear form which is the third derivative of $V(\xi)$, $\xi \in \mathbf{R}^d$ and. Let $\Psi(y, z, \phi(\cdot))$ be the solution to the equation

$$(2.20) \quad \begin{aligned} L\Psi &\equiv [d^* d + \nabla_y^* V''(\nabla \phi(y)) \nabla_y + \nabla_z^* V''(\nabla \phi(z)) \nabla_z + 2m^2] \Psi(y, z, \phi(\cdot)) \\ &= [\Phi_1(z, \phi(\cdot)) \nabla_y^* h_2(y) + \Phi_2(z, \phi(\cdot)) \nabla_y^* h_1(y)] \quad y, z \in Q. \end{aligned}$$

It follows from (2.18), (2.19), (2.20) that

$$(2.21) \quad \begin{aligned} & \langle G_1(\phi(\cdot)) G_2(\phi(\cdot)) G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu} = \\ & - \sum_{y, z \in Q} \langle V'''(\nabla \phi(y)) [\nabla_y \nabla_z \Psi(y, z, \phi(\cdot)), \nabla_y \Phi_3(y, \phi(\cdot))] \rangle_{\Omega_Q, m, x, \mu} \delta(y - z). \end{aligned}$$

Hence we obtain the inequality

$$(2.22) \quad |\langle G_1(\phi(\cdot)) G_2(\phi(\cdot)) G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu}| \leq \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)| \left\{ \sum_{y, z \in Q} \langle |\nabla_y \nabla_z \Psi(y, z, \phi(\cdot))|^2 \rangle_{\Omega_Q, m, x, \mu} \right\}^{1/2} \left\{ \sum_{y \in Q} \langle |\nabla_y \Phi_3(y, \phi(\cdot))|^2 \rangle_{\Omega_Q, m, x, \mu} \right\}^{1/2}.$$

From (2.17) the second term in curly braces on the RHS of (2.22) is bounded by $\|h_3\|/\lambda$. This follows from the fact that the norm of the operator

$$(2.23) \quad \nabla_y [d^* d + \nabla_y^* V''(\nabla \phi(y)) \nabla_y + m^2]^{-1} \nabla_y^*$$

is bounded uniformly for $m > 0$. Note that ∇ appears symmetrically in this expression so that quadratic form bounds apply. To estimate the third moment in terms of the L^2 norms of the $h_j(\cdot)$, $j = 1, 2, 3$ we need to bound the first term in curly braces.

Lemma 2.2. *Let Ψ be given by (2.20) and set*

$$(2.24) \quad \Phi = [\Phi_1(z, \phi(\cdot)) h_2(y) + \Phi_2(z, \phi(\cdot)) h_1(y)].$$

Then denoting expectation on Ω_Q by $\langle \cdot \rangle$, there is a constant $C(\lambda, \Lambda)$ depending only on the constants in (1.1) such that

$$(2.25) \quad \sum_{y, z \in Q} \langle |\nabla_y \nabla_z \Psi(y, z, \phi(\cdot))|^2 \rangle \leq C(\lambda, \Lambda) \sum_{y, z \in Q} \langle |\nabla_z \Phi(y, z, \phi(\cdot))|^2 \rangle,$$

provided $\lambda/\Lambda > 1/2$.

Proof. The ∇_z must be transferred to Φ . For this reason we introduce an elliptic system symmetric with respect to permutation of z and y , which enables us to use standard quadratic form methods to bound $\nabla_y \nabla_z \Psi(y, z, \phi(\cdot))$. From (1.1) we have that $V''(\xi) = \Lambda[I_d - \mathbf{b}(\xi)]$, $\xi \in \mathbf{R}^d$, where $\mathbf{b}(\cdot)$ satisfies the quadratic form inequality $0 \leq \mathbf{b}(\cdot) \leq (1 - \lambda/\Lambda)I_d$. We consider the system

$$(2.26) \quad \left\{ \begin{aligned} & [d^* d + \nabla_y^* V''(\nabla \phi(y)) \nabla_y + \Lambda \nabla_z^* \nabla_z + 2m^2] \Psi_1(y, z, \phi(\cdot)) \\ & - \Lambda \nabla_y^* \mathbf{b}(\nabla \phi(y)) \nabla_y \Psi_2(y, z, \phi(\cdot)) \end{aligned} \right\} = \nabla_y^* \Phi(y, z, \phi(\cdot)), \quad y, z \in Q,$$

$$\left\{ \begin{aligned} & [d^* d + \nabla_z^* V''(\nabla \phi(z)) \nabla_z + \Lambda \nabla_y^* \nabla_y + 2m^2] \Psi_2(y, z, \phi(\cdot)) \\ & - \Lambda \nabla_z^* \mathbf{b}(\nabla \phi(z)) \nabla_z \Psi_1(y, z, \phi(\cdot)) \end{aligned} \right\} = 0, \quad y, z \in Q.$$

Note that in the first equation of (2.26) the operator ∇_z commutes with the operator in the curly braces, and in the second equation the operator ∇_y similarly commutes. By adding the two equations we see that $L\Psi = L\Psi_1 + L\Psi_2 = \nabla_y^* \Phi$ as in (2.20) and $\Psi = \Psi_1 + \Psi_2$. We multiply the first equation of (2.26) by $\Psi_1(y, z, \phi(\cdot)) \nabla_z^* \nabla_z$, the second by $\Psi_2(y, z, \phi(\cdot)) \nabla_y^* \nabla_y$ and add, then sum over $y, z \in Q$ and take the expectation. This yields a quadratic form inequality on $\nabla_y \nabla_z \Psi_i$, $i = 1, 2$

$$(2.27) \quad \sum_{y, z \in Q} \langle \{ \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \}^* V''(\nabla \phi(y)) \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) + \\ \{ \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) \}^* V''(\nabla \phi(z)) \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) \rangle \\ - \Lambda \sum_{y, z \in Q} \langle \{ \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \}^* \mathbf{b}(\nabla \phi(y)) \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) + \\ \{ \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) \}^* \mathbf{b}(\nabla \phi(z)) \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \rangle \\ \leq \sum_{y, z \in Q} \langle \{ \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \}^* \nabla_z \Phi(y, z, \phi(\cdot)) \rangle .$$

Note that we have dropped positive terms such as those involving $d^* d + m^2$ and that through exchange of derivatives $\nabla_y \Phi$ in (2.26) has been replaced by $\nabla_z \Phi$. Assuming now that $\lambda/\Lambda > 1/2$ in (2.27), we apply the Schwarz inequality to obtain

$$(2.28) \quad \sum_{y, z \in Q} \langle \sum_{j=1,2} \{ \nabla_z \nabla_y \Psi_j(y, z, \phi(\cdot)) \}^* \nabla_z \nabla_y \Psi_j(y, z, \phi(\cdot)) \rangle \leq \\ (2\lambda - \Lambda)^{-2} \sum_{y, z \in Q} \langle |\nabla_z \Phi(y, z, \phi(\cdot))|^2 \rangle .$$

□

Proof of Theorem 2.1. From (2.17) it follows that

$$(2.29) \quad \sum_{y, z \in Q} \langle |\nabla_z \Phi(y, z, \phi(\cdot))|^2 \rangle \leq [2\|h_1\| \|h_2\|/\lambda]^2 .$$

Now Lemma 2.2 and (2.22) imply that

$$(2.30) \quad |\langle G_1(\phi(\cdot)) G_2(\phi(\cdot)) G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu}| \leq \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)| C(\lambda, \Lambda) \|h_1\| \|h_2\| \|h_3\|$$

provided $\lambda/\Lambda > 1/2$. The result follows from (2.30) and Proposition 2.1, 2.2. □

We show how to generate the solution to (2.20) by means of a converging perturbation expansion in $\mathbf{b}(\cdot)$. This expansion yields an alternative proof of Lemma 2.2, and we shall use it in §4 to prove Theorem 1.1. Let T_1, T_2 be defined by

$$(2.31) \quad \begin{aligned} T_1 &\equiv \nabla_y [d^*d/\Lambda + \nabla_y^* \nabla_y + \nabla_z^* \nabla_z + 2m^2/\Lambda]^{-1} \nabla_y^*, \\ T_2 &\equiv \nabla_z [d^*d/\Lambda + \nabla_y^* \nabla_y + \nabla_z^* \nabla_z + 2m^2/\Lambda]^{-1} \nabla_z^*, \end{aligned}$$

We define a matrix $\mathbf{B}(\cdot)$ by

$$(2.32) \quad \mathbf{B}(y, z, \phi(\cdot)) = \begin{bmatrix} \mathbf{b}(\nabla\phi(y)) & \mathbf{b}(\nabla\phi(y)) \\ \mathbf{b}(\nabla\phi(z)) & \mathbf{b}(\nabla\phi(z)) \end{bmatrix}.$$

Note that in (2.32) the $d \times d$ matrix $\mathbf{b}(\nabla\phi(y))$ acts only on the y entries of $\Phi_j(y, z, \phi(\cdot))$, $j = 1, 2$. Then one can check that (2.26) is equivalent to the system

$$(2.33) \quad \begin{bmatrix} \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \\ \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \nabla_z \Phi(y, z, \phi(\cdot)) \\ 0 \end{bmatrix} \\ + \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \mathbf{B}(y, z, \phi(\cdot)) \begin{bmatrix} \nabla_z \nabla_y \Psi_1(y, z, \phi(\cdot)) \\ \nabla_z \nabla_y \Psi_2(y, z, \phi(\cdot)) \end{bmatrix}.$$

It is evident that the operator $\mathbf{B}(\cdot)$ of (2.32) has L^2 norm less than $2(1 - \lambda/\Lambda)$. Since the operators T_1, T_2 defined by (2.31) have norm less than 1, the Neumann series for the solution of (2.33) converges in $L^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ if $\lambda/\Lambda > 1/2$.

The solution of (2.17) can also be generated by a converging perturbation expansion in $\mathbf{b}(\cdot)$ in the usual way. Thus let $\Psi(y, \phi(\cdot))$ be the solution to the equation

$$(2.34) \quad [d^*d/\Lambda + \nabla^* \nabla + m^2/\Lambda] \Psi(y, \phi(\cdot)) = \nabla^* \Phi(y, \phi(\cdot)), \quad y \in Q.$$

Then we write $\nabla \Psi = T \Phi$ which defines the operator T . Equation (2.17) is equivalent to

$$(2.35) \quad \nabla \Psi(y, \phi(\cdot)) = \Lambda^{-1} T \Phi(y, \phi(\cdot)) + T[\mathbf{b}(\nabla\phi(y)) \nabla \Psi(y, \phi(\cdot))],$$

with $\Phi(y, \phi(\cdot)) = h_j(y)$, $y \in Q$. Since the operator T has norm which does not exceed 1, the Neumann series for the solution of (2.35) converges for any $\lambda/\Lambda > 0$ in $L^2(Q \times \Omega_Q, \mathbf{R}^d)$ with measure (2.2) on Ω_Q .

3. WEIGHTED NORM INEQUALITIES ON ℓ^2 SPACES

In this section we prove the weighted norm inequalities on ℓ^2 spaces which we shall need to prove Theorem 1.1. This section is independent of the previous one. For a positive periodic function $w : Q \rightarrow \mathbf{R}$ the associated weighted space $\ell_w^2(Q, \mathbf{R}^d)$ is all periodic functions $h : Q \rightarrow \mathbf{R}^d$ with norm $\|h\|_w$ defined by

$$(3.1) \quad \|h\|_w^2 = \sum_{y \in Q} w(y) |h(y)|^2.$$

Define the Green's function on \mathbf{Z}^d by

$$(3.2) \quad [\nabla^* \nabla + \rho] G_\rho(y) = \delta(y), \quad y \in \mathbf{Z}^d.$$

Thus we have that

$$(3.3) \quad |\nabla G_\rho(y)| \leq C/[1 + |y|]^{d-1}, \quad |\nabla \nabla^* G_\rho(y)| \leq C/[1 + |y|]^d, \\ |\nabla \nabla \nabla^* G_\rho(y)| \leq C/[1 + |y|]^{d+1}, \quad y \in \mathbf{Z}^d, \quad \rho > 0,$$

for some constant C depending only on d . The corresponding periodic Green's function for the cube Q with side of length L is

$$(3.4) \quad G_{\rho,Q}(y) = \sum_{y' \in \mathbf{Z}^d} G_{\rho}(y + Ly') .$$

In order to estimate the periodic Green's function we need in addition to (3.3) the inequalities

$$(3.5) \quad \left| \sum_{y' \in \mathbf{Z}^d - \{0\}} \nabla G_{\rho}(y + Ly') \right| \leq C/L^{d-1}, \quad \left| \sum_{y' \in \mathbf{Z}^d - \{0\}} \nabla \nabla^* G_{\rho}(y + Ly') \right| \leq C/L^d, \quad y \in Q, \rho > 0,$$

which hold for a constant C depending only on d . Note that the sums in (3.5) are not absolutely convergent uniformly for $\rho > 0$. The Calderon-Zygmund operator T_{ρ} in a periodic domain is explicitly given by the formula

$$(3.6) \quad T_{\rho}h(y) = \sum_{y' \in Q} \nabla \nabla^* G_{\rho,Q}(y - y')h(y') ,$$

where $G_{\rho,Q}(\cdot)$ is the function (3.4). The inequalities (3.3), (3.5) therefore yield an estimate on the kernel of T_{ρ} , which is independent of $\rho > 0$. The basic proposition for this section may be stated as follows.

Proposition 3.1. *Let $w : Q \rightarrow \mathbf{R}$ be given by $w(y) = [1 + |y|]^{\alpha}$, $y \in Q$, where $|\alpha| < d$. Then T_{ρ} is bounded on $\ell_w^2(Q, \mathbf{R}^d)$ for $\rho > 0$, and $\|T_{\rho}\|_w \leq 1 + C_{\varepsilon}|\alpha|$ where the constant C_{ε} depends only on d and any $\varepsilon < d - |\alpha|$.*

Proof. Adapting the methods of Chapter V of [15] to the periodic lattice, it is clear in view of the inequalities (3.3), (3.5) that the result holds for $d/2 \leq |\alpha| < d$. Now we apply [16] as in the recent work of Pattakos and Volberg [12] to obtain the inequality for $\|T_{\rho}\|_w$ when $|\alpha|$ is small. \square

Next we consider operators on weighted function spaces of two variables. For a positive periodic function $w : Q \times Q \rightarrow \mathbf{R}$ the associated weighted space $\ell_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ is all periodic functions $h : Q \times Q \rightarrow \mathbf{R}^d \times \mathbf{R}^d$ with norm $\|h\|_w$ defined by

$$(3.7) \quad \|h\|_w^2 = \sum_{(y,z) \in Q \times Q} w(y,z) |h(y,z)|^2 .$$

Let $T_{\rho} \otimes I$ be the operator on $\ell_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ which acts by the operator T_{ρ} defined by (3.6) on the y variable of a function $h(y,z)$ and by the identity on the z variable.

Proposition 3.2. *Let $w : Q \times Q \rightarrow \mathbf{R}$ be given by $w(y,z) = [1 + |y|]^{\alpha}[1 + \gamma(z,y)]^{\beta}$, $(y,z) \in Q \times Q$, where $\gamma(z,y)$ is the shortest distance from z to y on the periodic cube Q . Then if $|\alpha| < d$, $|\beta| < d$ the operator $T_{\rho} \otimes I$ is bounded on $\ell_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ for $\rho > 0$, and $\|T_{\rho} \otimes I\|_w \leq 1 + C_{\varepsilon}[|\alpha| + |\beta|]$ where the constant C_{ε} depends only on d and any $\varepsilon < \min\{[d - |\alpha|], [d - |\beta|]\}$.*

Proof. Same as for Proposition 3.1. \square

For $\rho > 0$ let $T_{1,\rho}$ be the operator on periodic functions $h : Q \times Q \rightarrow \mathbf{R}^d \times \mathbf{R}^d$ defined by

$$(3.8) \quad T_{1,\rho} \equiv \nabla_y [\nabla_y^* \nabla_y + \nabla_z^* \nabla_z + \rho]^{-1} \nabla_y^* .$$

Let $G_\rho(y, z)$, $y, z \in \mathbf{Z}^d$, be the Green's function for the discrete Laplacian on \mathbf{Z}^{2d} defined as in (3.2), and $G_{\rho, Q \times Q}(y, z)$, $y, z \in Q$, be the corresponding periodic Green's function for the cube $Q \times Q$ defined as in (3.4). The operator $T_{1, \rho}$ is explicitly given by the formula

$$(3.9) \quad T_{1, \rho} h(y, z) = \sum_{(y', z') \in Q \times Q} \nabla_y \nabla_y^* G_{\rho, Q \times Q}(y - y', z - z') h(y', z').$$

In (3.9) the row vector $\nabla_y^* G_{\rho, Q \times Q}(y - y', z - z')$ acts on the y' array of the double array column vector $h(y', z')$.

Proposition 3.3. *Let $w : Q \times Q \rightarrow \mathbf{R}$ be given by $w(y, z) = [1 + |y|]^\alpha [1 + \gamma(z, y)]^\beta$ or $w(y, z) = [1 + |z|]^\alpha [1 + \gamma(z, y)]^\beta$, $(y, z) \in Q \times Q$, where $|\alpha|, |\beta| < d$. Then $T_{1, \rho}$ is bounded on $\ell_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ for $\rho > 0$, and $\|T_{1, \rho}\|_w \leq 1 + C_\varepsilon[|\alpha| + |\beta|]$ where the constant C_ε depends only on d and any $\varepsilon < \min\{[d - |\alpha|], [d - |\beta|]\}$.*

Proof. Same as for Proposition 3.1. \square

4. WEIGHTED L^2 THEORY

Our goal in this section will be to extend Theorem 2.1 to allow the functions $h_j : Q \rightarrow \mathbf{R}^d$, $j = 1, 2, 3$, to lie in certain *weighted* ℓ^2 spaces. This is needed for the proof of Theorem 1.1. In order to carry this out we define weighted versions of the L^2 spaces of §2. Thus for a periodic weight $w : Q \rightarrow \mathbf{R}$ the associated weighted space $L_w^2(Q \times \Omega_Q, \mathbf{R}^d)$ is the space of all periodic measurable functions $\Phi : Q \times \Omega_Q \rightarrow \mathbf{R}^d$ with finite norm $\|\Phi\|_w$ given by

$$(4.1) \quad \|\Phi\|_w^2 = \sum_{y \in Q} w(y) \langle |\Phi(y, \phi(\cdot))|^2 \rangle_{\Omega_Q, m, x, \mu}.$$

Letting T be the operator defined by (2.34), it follows from the spectral decomposition theorem for d^*d , that T is bounded on $L_w^2(Q \times \Omega_Q, \mathbf{R}^d)$ since the operator T_ρ of (3.6) is bounded on $\ell_w^2(Q, \mathbf{R}^d)$ for all $\rho \geq m^2/\Lambda$. Furthermore one has the inequality

$$(4.2) \quad \|T\|_w \leq \sup_{\rho \geq m^2/\Lambda} \|T_\rho\|_w.$$

Similarly one can define for a periodic weight $w : Q \times Q \rightarrow \mathbf{R}$ the weighted space $L_w^2(Q \times Q \times \Omega_Q, \mathbf{R}^d)$ as the space of all periodic measurable functions $\Phi : Q \times Q \times \Omega_Q \rightarrow \mathbf{R}^d$ with finite norm $\|\Phi\|_w$ given by

$$(4.3) \quad \|\Phi\|_w^2 = \sum_{(y, z) \in Q \times Q} w(y, z) \langle |\Phi(y, z, \phi(\cdot))|^2 \rangle_{\Omega_Q, m, x, \mu}.$$

The operator T_1 defined by (2.31) is bounded on $L_w^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ since the operator $T_{1, \rho}$ of (3.9) is bounded on $\ell_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ for all $\rho \geq 2m^2/\Lambda$ and

$$(4.4) \quad \|T_1\|_w \leq \sup_{\rho \geq 2m^2/\Lambda} \|T_{1, \rho}\|_w.$$

Finally we define the weighted space $L_w^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ with norm as in (4.3). Then by the spectral decomposition theorem the operator $T \otimes I$ is bounded

on $L_w^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ since $T_\rho \otimes I$ is bounded on $l_w^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$ for all $\rho \geq m^2/\Lambda$. In that case one has the inequality

$$(4.5) \quad \|T \otimes I\|_w \leq \sup_{\rho \geq m^2/\Lambda} \|T_\rho \otimes I\|_w.$$

We can now state a weighted version of Theorem 2.1. For $\alpha, \beta \in \mathbf{R}$ let $w_\alpha : Q \rightarrow \mathbf{R}$, $w_{\alpha, \beta} : Q \times Q \rightarrow \mathbf{R}$ be the weights $w_\alpha(y) = [1 + |y|]^\alpha$, $y \in Q$, and $w_{\alpha, \beta}(y, z) = [1 + |y|]^\alpha [1 + \gamma(y, z)]^\beta$, $y, z \in Q$, where $\gamma(y, z)$ is the distance from y to z in the periodic cube Q .

Theorem 4.1. *Suppose Q is a periodic cube in \mathbf{Z}^d for some $d \geq 1$ and that $h_j : Q \rightarrow \mathbf{R}^d$, $j = 1, 2, 3$, have the property that $h_3 \in \ell_{w_{-\alpha}}^2(Q, \mathbf{R}^d)$ and both $h_1 \otimes h_2$, $h_2 \otimes h_1$ are in $\ell_{w_{\alpha, \beta}}^2(Q \times Q, \mathbf{R}^d \times \mathbf{R}^d)$. Then for $|\alpha|$, $|\beta|$ sufficiently small depending only on $\lambda/\Lambda > 1/2$, there is a positive constant $C(\lambda, \Lambda)$ depending only on λ, Λ , such that for any $x \in Q$, $\mu \in \mathbf{R}$,*

$$(4.6) \quad \left| \left\langle \prod_{j=1}^3 [h_j, \omega] - \langle (h_j, \omega) \rangle_{\Omega_Q, x, m, \mu} \right\rangle_{\Omega_Q, x, m, \mu} \right| \leq C(\lambda, \Lambda) \left[\|h_1 \otimes h_2\|_{w_{\alpha, \beta}} + \|h_2 \otimes h_1\|_{w_{\alpha, \beta}} \right] \|h_3\|_{w_{-\alpha}} \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)|.$$

Remark: For the proof of Theorem 1.1 we will choose $\beta < -\alpha$ with $\alpha > 0$ small.

Proof. We first consider the function $\Phi_3(y, \phi(\cdot))$ defined by (2.17), whose gradient $\nabla \Phi_3(y, \phi(\cdot))$ is given by the Neumann series for the solution of (2.35). In view of (4.2) and Proposition 3.1, the series converges in $L_{w_{-\alpha}}^2(Q \times \Omega_Q, \mathbf{R}^d)$ provided $|\alpha|$ is sufficiently small, depending only on $\lambda/\Lambda > 0$, and $\|\nabla \Phi_3(\cdot, \phi(\cdot))\|_{w_{-\alpha}} \leq C(\lambda, \Lambda) \|h_3\|_{w_{-\alpha}}$.

Next we consider the function $\Phi : Q \times Q \rightarrow \mathbf{R}^d \times \mathbf{R}^d$ defined by $\Phi(y, z, \phi(\cdot)) = \nabla \Phi_1(z, \phi(\cdot)) h_2(y) + \nabla \Phi_2(z, \phi(\cdot)) h_1(y)$, $y, z \in Q$, where the $\Phi_j(\cdot, \phi(\cdot))$, $j = 1, 2$ are solutions of (2.17). It follows from (4.5) and Proposition 3.2 that Φ is in $L_{w_{\alpha, \beta}}^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ if $|\alpha| + |\beta|$ is sufficiently small, depending only on $\lambda/\Lambda > 0$, and $\|\Phi\|_{w_{\alpha, \beta}} \leq C(\lambda, \Lambda) [\|h_1 \otimes h_2\|_{w_{\alpha, \beta}} + \|h_2 \otimes h_1\|_{w_{\alpha, \beta}}]$. For $\Phi \in L_{w_{\alpha, \beta}}^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ we can generate the solution to (2.20) by means of the perturbation expansion generated by (2.33). It follows then from (4.4) and Proposition 3.3 that $\nabla_y \nabla_z \Psi(y, z, \phi(\cdot))$, $y, z \in Q$, is in $L_{w_{\alpha, \beta}}^2(Q \times Q \times \Omega_Q, \mathbf{R}^d \times \mathbf{R}^d)$ if $|\alpha| + |\beta|$ is sufficiently small, depending only on λ/Λ with $1/2 < \lambda/\Lambda \leq 1$, and $\|\nabla \nabla \Psi\|_{w_{\alpha, \beta}} \leq C(\lambda, \Lambda) \|\Phi\|_{w_{\alpha, \beta}}$.

To complete the proof of (4.6) we use the representation (2.21). Using the Schwarz inequality as in (2.22) we conclude that

$$(4.7) \quad \begin{aligned} & \left| \langle G_1(\phi(\cdot)) G_2(\phi(\cdot)) G_3(\phi(\cdot)) \rangle_{\Omega_Q, m, x, \mu} \right| \leq \\ & \quad \|\nabla \nabla \Psi\|_{w_{\alpha, \beta}} \|\nabla \Phi_3\|_{w_{-\alpha}} \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)| \\ & \leq C(\lambda, \Lambda) \left[\|h_1 \otimes h_2\|_{w_{\alpha, \beta}} + \|h_2 \otimes h_1\|_{w_{\alpha, \beta}} \right] \|h_3\|_{w_{-\alpha}} \sup_{\xi \in \mathbf{R}^d} |V'''(\xi)|. \end{aligned}$$

□

Proof of Theorem 1.1. By Proposition 2.1 and 2.2 it will be sufficient to obtain an estimate for $|\langle [X - \langle X \rangle_{\Omega_{Q,x,m,\mu}}]^3 \rangle_{\Omega_{Q,x,m,\mu}}|$ with $X = \phi(0) - \phi(x)$, which is uniform as $Q \rightarrow \mathbf{Z}^d$ and $m \rightarrow 0$.

One can see from (3.3), (3.5) that there is a periodic function $h_Q : Q \rightarrow \mathbf{R}^d$ satisfying

$$(4.8) \quad h_Q(y) = \lim_{\rho \rightarrow 0} \nabla G_{\rho,Q}(y), \quad |h_Q(y)| \leq C/[1 + |y|]^{d-1}, \quad y \in Q,$$

for a constant C depending only on d .

We may write $\phi(0) - \phi(x)$ in terms of the function $h_Q(\cdot)$ by means of (3.2). Thus we have that

$$(4.9) \quad \phi(0) - \phi(x) = (\nabla G_{\rho,Q}, \omega) - (\tau_x \nabla G_{\rho,Q}, \omega) + (\rho[(G_{\rho,Q} - \tau_x G_{\rho,Q}], \phi),$$

where τ_x denotes translate of a function by x . From the first inequalities of (3.3), (3.5) it follows that $\lim_{\rho \rightarrow 0} (\rho[(G_{\rho,Q} - \tau_x G_{\rho,Q}], \phi) = 0$, whence (4.8), (4.9) imply that

$$(4.10) \quad \phi(0) - \phi(x) = (h_Q, \omega) - (\tau_x h_Q, \omega).$$

In order to prove Theorem 1.1 it will therefore be sufficient for us to apply Theorem 4.1 for $h_1 = h_2 = h_Q$ and $h_3 = h_Q$ or $h_3 = \tau_x h_Q$. One easily sees that for $d = 2$ and $0 < \alpha < d$, there is a constant C_α depending only on α such that

$$(4.11) \quad \|\tau_x h_Q\|_{w_{-\alpha}} \leq C_\alpha/[1 + |x|]^\alpha, \quad x \in Q.$$

Similarly one has that for $d = 2$ and $0 < \alpha < d$, $-d < \beta < -\alpha$, there is a constant $C_{\alpha,\beta}$ depending only on α, β such that

$$(4.12) \quad \|h_Q \otimes h_Q\|_{w_{\alpha,\beta}} \leq C_{\alpha,\beta}.$$

The inequality (1.7) follows from (4.11), (4.12) and Theorem 4.1. \square

Proof of Theorem 1.2. We observe that by translation invariance of the measure we only need to take $h_1 = h_2 = h_Q$, $h_3 = \tau_x h_Q$ in Theorem 4.1. \square

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